LIMIT THEOREMS FOR MARKED HAWKES PROCESSES WITH APPLICATION TO A RISK MODEL

DMYTRO KARABASH AND LINGJIONG ZHU

Abstract. This paper focuses on limit theorems for linear Hawkes processes with random marks. We prove a large deviation principle, which answers the question raised by Bordenave and Torrisi. A central limit theorem is also obtained. We conclude with an example of application in finance.

Contents

1. Introduction and Main Results

1.1. **Introduction.** We consider in this article a linear Hawkes process with random marks. Let N_t be a simple point process. N_t denotes the number of points on interval $[0,t)$ and $\mathcal{F}_t$ be the natural filtration up to time t. We assume that $N(-\infty,0]=0$. At time t, the point process has $\mathcal{F}_t$-predictable intensity λ_t, where

$$\lambda_t := \nu + Z_t, \quad Z_t := \sum_{\tau_i<t} h(t-\tau_i, a_i), \tag{1.1}$$

where $\nu>0$ and $(\tau_i)_{i\geq 1}$ are arrival times of the points and $(a_i)_{i\geq 1}$ are i.i.d. random marks, where a_i is independent of previous arrival times τ_j, $j\leq i$. Let us assume that a_i has a common distribution $q(da)$ on space $\mathbb{X}$. Here, $h(\cdot,\cdot):\mathbb{R}^+\times\mathbb{X}\to\mathbb{R}^+$ is

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integrable, i.e. $\int_0^\infty \int_{\mathbb{X}} h(t,a)q(da)dt < \infty$. Let $H(a) := \int_0^\infty h(t,a)dt$ for any $a \in \mathbb{X}$. We also assume that

$$\int_{\mathbb{X}} H(a)q(da) < 1. \tag{1.2}$$

Let $\mathbb{P}^q$ denote the probability measure for a_i with the common law $q(da)$. Under assumption (1.2), it is well known that there exists a unique stationary version of linear marked Hawkes process satisfying the dynamics (1.1) and by ergodic theorem, a law of large numbers holds,

$$\lim_{t\to\infty} \frac{N_t}{t} = \frac{\nu}{1-\mathbb{E}^q[H(a)]}. \tag{1.3}$$

This paper is organized as the following. In Section 1.2, we will review some results about the limit theorems for unmarked Hawkes processes. In Section 1.3, we will introduce the main results of this paper, i.e. the central limit theorem and large deviation principle for linear marked Hawkes processes. The proof of the central limit theorem will be given in Section 2 and the proof of the large deviation principle will be given in Section 3. Finally, we will discuss an application of our results to a risk model in finance in Section 4.

1.2. **Limit Theorems for Unmarked Hawkes Processes.** Most of the literatures about Hawkes process consider the unmarked case, i.e. with intensity

$$\lambda_t := \lambda\left(\sum_{\tau<t} h(t-\tau)\right), \tag{1.4}$$

where $h(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ is integrable and $\|h\|_{L^1} < 1$ and $\lambda(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ is locally integrable and left continuous.

When $\lambda(\cdot)$ is linear, the Hawkes process is said to be linear and it is nonlinear otherwise. The stability results for both linear and nonlinear Hawkes processes are known. For the linear case, we refer to Daley and Vere-Jones [4]. For the nonlinear case, Brémaud and Massoulié [3] proved the stability results for α-Lipschitz $\lambda(\cdot)$ such that $\alpha\|h\|_{L^1} < 1$. Karabash [8] obtained stability results for certain non-Lipschitz $\lambda(\cdot)$ and discontinuous $\lambda(\cdot)$.

The limit theorems for both linear and nonlinear Hawkes processes are also known.

For the linear Hawkes process, assume $\lambda(z) = \nu + z$, for some $\nu > 0$ and $\|h\|_{L^1} < 1$, it has a very nice immigration-birth representation, see for example Hawkes and Oakes [7]. For linear Hawkes process, limit theorems are very well understood. There is the law of large numbers (see for instance Daley and Vere-Jones [4]), i.e.

$$\frac{N_t}{t} \to \frac{\nu}{1-\|h\|_{L^1}}, \quad \text{as } t \to \infty \text{ a.s.} \tag{1.5}$$

Moreover, Bordenave and Torrisi [2] proved a large deviation principle for $(\frac{N_t}{t} \in \cdot)$ with the rate function

$$I(x) = \begin{cases} x\log\left(\frac{x}{\nu+x\|h\|_{L^1}}\right) - x + x\|h\|_{L^1} + \nu & \text{if } x \in [0,\infty) \\ +\infty & \text{otherwise} \end{cases}. \tag{1.6}$$

Once you have the large deviation principle, you can also study some risk processes in finance. (See Stabile and Torrisi [9].)

Recently, Bacry et al. [1] proved a functional central limit theorem for linear multivariate Hawkes process under certain assumptions which includes the linear Hawkes process as a special case and they proved that

$$\frac{N_{\cdot t} - \cdot \mu t}{\sqrt{t}} \to \sigma B(\cdot), \quad \text{as } t \to \infty, \tag{1.7}$$

weakly on $D[0,1]$ equipped with Skorokhod topology, where

$$\mu = \frac{\nu}{1 - \|h\|_{L^1}} \quad \text{and} \quad \sigma^2 = \frac{\nu}{(1 - \|h\|_{L^1})^3}. \tag{1.8}$$

Moderate deviation principle for linear Hawkes process is obtained in Zhu [15], which fills in the gap between central limit theorem and large deviation principle.

For nonlinear Hawkes process, a central limit theorem is obtained in Zhu [14]. In Bordenave and Torrisi [2], they raised two questions about large deviations for Hawkes processes. One question is about large deviations for nonlinear Hawkes process and the other is about large deviations for linear marked Hawkes process. Recently, Zhu [12] considered a special case for nonlinear Hawkes process when $h(\cdot)$ is exponential or sums of exponentials and proved the large deviations. In another paper, Zhu [13] proved a process-level, i.e. level-3 large deviation principle for nonlinear Hawkes process for general $h(\cdot)$ and hence by contraction principle, the level-1 large deviation principle for $(N_t/t \in \cdot)$. In this paper, we will prove the large deviations for linear marked Hawkes process and thus both questions raised in Bordenave and Torrisi [2] have been answered.

1.3. **Main Results.** Before we proceed, recall that a sequence $(P_n)_{n\in\mathbb{N}}$ of probability measures on a topological space X satisfies the large deviation principle (LDP) with rate function $I : X \to \mathbb{R}$ if I is non-negative, lower semicontinuous and for any measurable set A, we have

$$-\inf_{x\in A^o} I(x) \le \liminf_{n\to\infty} \frac{1}{n} \log P_n(A) \le \limsup_{n\to\infty} \frac{1}{n} \log P_n(A) \le -\inf_{x\in\overline{A}} I(x). \tag{1.9}$$

Here, A^o is the interior of A and $\overline{A}$ is its closure. See Dembo and Zeitouni [5] or Varadhan [11] for general background regarding large deviations and the applications. Also Varadhan [10] has an excellent survey article on this subject.

For a linear marked Hawkes process satisfying the dynamics (1.1), we prove the following large deviation principle in this article.

Theorem 1 (Large Deviation Principle)**.** *Assume the conditions* (1.2) *and*

$$\lim_{x\to\infty} \left\{ \int_{\mathbb{X}} e^{H(a)x} q(da) - x \right\} = \infty. \tag{1.10}$$

Then, we have $(N_t/t \in \cdot)$ satisfies a large deviation principle with rate function,

$$\Lambda(x) := \begin{cases} \inf_{\hat{q}} \left\{ x\mathbb{E}^{\hat{q}}[H(a)] + \nu - x + x\log\left(\frac{x}{x\mathbb{E}^{\hat{q}}[H(a)]+\nu}\right) + x\mathbb{E}^{\hat{q}}\left[\log\frac{d\hat{q}}{dq}\right]\right\} & x \ge 0 \\ +\infty & x < 0 \end{cases}$$

$$= \begin{cases} \theta_* x - \nu(x_* - 1) & x \ge 0 \\ +\infty & x < 0 \end{cases},$$

where the infimum of $\hat{q}$ is taken over $\mathcal{M}(\mathbb{X})$, the space of probability measures on $\mathbb{X}$ such that $\hat{q}$ is absolutely continuous w.r.t. q. Also, θ_* and x_* would satisfy the

following equations

$$\begin{cases} x_* = \mathbb{E}^q\left[e^{\theta_*+(x_*-1)H(a)}\right] \\ \frac{x}{\nu} = x_* + \frac{x}{\nu}\mathbb{E}^q\left[H(a)e^{\theta_*+(x_*-1)H(a)}\right] \end{cases}. \tag{1.11}$$

Theorem 2 (Central Limit Theorem)**.** *Assume* $\lim_{t\to\infty} t^{1/2}\int_t^\infty \mathbb{E}^q[h(s,a)]ds = 0$ *and the condition* (1.2) *holds. Then, we have*

$$\frac{N_t - \frac{\nu t}{1-\mathbb{E}^q[H(a)]}}{\sqrt{t}} \to N\left(0, \frac{\nu(1+Var^q[H(a)])}{(1-\mathbb{E}^q[H(a)])^3}\right), \tag{1.12}$$

in distribution as $t\to\infty$.

Remark 3. *Comparing Theorem 1 and Theorem 2 with* (1.6) *and* (1.7)*, it is easy to see that our results are consistent with the limit theorems for unmarked Hawkes process.*

2. Proof of Central Limit Theorem

Proof of Theorem 2. First, let us observe that

$$\begin{aligned}\int_0^t \lambda_s ds &= \nu t + \sum_{\tau_i<t}\int_{\tau_i}^t h(s-\tau_i,a_i)ds \\ &= \nu t + \sum_{\tau_i<t} H(a_i) - \mathcal{E}_t,\end{aligned} \tag{2.1}$$

where the error term $\mathcal{E}_t$ is given by

$$\mathcal{E}_t := \sum_{\tau_i<t}\int_t^\infty h(s-\tau_i,a_i)ds. \tag{2.2}$$

Therefore, we get

$$\begin{aligned}\frac{N_t - \int_0^t \lambda_s ds}{\sqrt{t}} &= \frac{N_t - \nu t - \sum_{\tau_i<t} H(a_i)}{\sqrt{t}} + \frac{\mathcal{E}_t}{\sqrt{t}} \\ &= (1-\mathbb{E}^q[H(a)])\frac{N_t-\mu t}{\sqrt{t}} + \frac{\mathbb{E}^q[H(a)]N_t - \sum_{\tau_i<t} H(a_i)}{\sqrt{t}} + \frac{\mathcal{E}_t}{\sqrt{t}},\end{aligned} \tag{2.3}$$

where $\mu := \frac{\nu}{1-\mathbb{E}^q[H(a)]}$. Rearranging the terms in (2.3), we get

$$\frac{N_t-\mu t}{\sqrt{t}} = \frac{1}{1-\mathbb{E}^q[H(a)]}\left[\frac{N_t - \int_0^t \lambda_s ds}{\sqrt{t}} + \frac{\sum_{\tau_i<t}(H(a_i) - \mathbb{E}^q[H(a)])}{\sqrt{t}} - \frac{\mathcal{E}_t}{\sqrt{t}}\right]. \tag{2.4}$$

It is easy to check that $\frac{\mathcal{E}_t}{\sqrt{t}} \to 0$ in probability as $t\to\infty$. To see this, first notice that $\mathbb{E}[\lambda_t] \leq \frac{\nu}{1-\mathbb{E}^q[H(a)]}$ uniformly in t. Let $g(t,a) := \int_t^\infty h(s,a)ds$. We have $\mathcal{E}_t = \sum_{\tau_i<t} g(t-\tau_i,a_i)$ and thus

$$\begin{aligned}\mathbb{E}[\mathcal{E}_t] &= \int_0^t \int_{\mathbb{X}} g(t-s,a)q(da)\mathbb{E}[\lambda_s]ds \\ &\leq \frac{\nu}{1-\mathbb{E}^q[H(a)]}\int_0^t\int_{\mathbb{X}} g(t-s,a)q(da)ds \\ &= \frac{\nu}{1-\mathbb{E}^q[H(a)]}\int_0^t \mathbb{E}^q[g(s,a)]ds.\end{aligned} \tag{2.5}$$

Hence, by L'Hôpital's rule,

$$\begin{aligned}\lim_{t\to\infty}\frac{1}{t^{1/2}}\int_0^t \mathbb{E}^q[g(s,a)]ds &= \lim_{t\to\infty}\frac{\mathbb{E}^q[g(t,a)]}{\frac{1}{2}t^{-1/2}} \\ &= \lim_{t\to\infty}2t^{1/2}\int_t^\infty \mathbb{E}^q[h(s,a)]ds = 0.\end{aligned} \tag{2.6}$$

Hence, $\frac{\mathcal{E}_t}{\sqrt{t}} \to 0$ in probability as $t\to\infty$.

Furthermore, $M_1(t) := N_t - \int_0^t \lambda_s ds$ and $M_2(t) := \sum_{\tau_i<t}(H(a_i) - \mathbb{E}^q[H(a)])$ are both martingales.

Moreover, since $\int_0^t \lambda_s ds$ is of finite variation, the quadratic variation of $M_1(t) + M_2(t)$ is the same as the quadratic variation of $N_t + M_2(t)$. And notice that $N_t + M_2(t) = \sum_{\tau_i<t}(1 + H(a_i) - \mathbb{E}^q[H(a)])$ which has quadratic variation

$$\sum_{\tau_i<t}(1 + H(a_i) - \mathbb{E}^q[H(a)])^2. \tag{2.7}$$

By standard law of large numbers, we have

$$\begin{aligned}\frac{1}{t}\sum_{\tau_i<t}(1 + H(a_i) - \mathbb{E}^q[H(a)]) &= \frac{N_t}{t}\cdot\frac{1}{N_t}\sum_{\tau_i<t}(1 + H(a_i) - \mathbb{E}^q[H(a)])^2 \\ &\to \frac{\nu}{1-\mathbb{E}^q[H(a)]}\cdot\mathbb{E}^q\left[(1 + H(a) - \mathbb{E}^q[H(a)])^2\right] \\ &= \frac{\nu(1+\mathrm{Var}^q[H(a)])}{1-\mathbb{E}^q[H(a)]},\end{aligned} \tag{2.8}$$

a.s. as $t\to\infty$. By a standard martingale central limit theorem, we conclude that

$$\frac{N_t - \frac{\nu t}{1-\mathbb{E}^q[H(a)]}}{\sqrt{t}} \to N\left(0, \frac{\nu(1+\mathrm{Var}^q[H(a)])}{(1-\mathbb{E}^q[H(a)])^3}\right), \tag{2.9}$$

in distribution as $t\to\infty$. □

3. Proof of Large Deviation Principle

3.1. **Limit of Logarithmic Moment Generating Function.** In this subsection, we prove the existence of the limit of the logarithmic moment generating function $\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}[e^{\theta N_t}]$ and give a variational formula and a more explicit formula for this limit.

Theorem 4. *The limit of logarithmic moment generating function $\Gamma(\theta)$ exists and we have*

$$\Gamma(\theta) = \lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}[e^{\theta N_t}] = \begin{cases}\nu(f(\theta)-1) & \text{if } \theta\in(-\infty,\theta_c] \\ +\infty & \text{otherwise}\end{cases}, \tag{3.1}$$

where $f(\theta)$ is the minimal solution to $x = \int_{\mathbb{X}} e^{\theta + H(a)(x-1)}q(da)$ and

$$\theta_c = -\log\int_{\mathbb{X}} H(a)e^{H(a)(x_c-1)}q(da) > 0, \tag{3.2}$$

where $x_c > 1$ satisfies the equation $x\int_{\mathbb{X}} H(a)e^{H(a)(x-1)}q(da) = \int_{\mathbb{X}} e^{H(a)(x-1)}q(da)$.

We will break the proof of Theorem 4 into the proof of lower bound, i.e. Lemma 6 and the proof of upper bound, i.e. Lemma 7.

Before we proceed, let us first prove Lemma 5, which will be repeatedly used in the proofs in our paper.

Lemma 5. *Consider a linear marked Hawkes process with intensity*

$$\lambda_t := \alpha + \beta Z_t := \alpha + \beta \sum_{\tau_i < t} h(t-\tau_i, a_i), \tag{3.3}$$

and $\beta\mathbb{E}^q[H(a)] < 1$, *where* a_i *are i.i.d. random marks with the common law* $q(da)$ *independent of the previous arrival times, then there exists a unique invariant measure* π *for* Z_t *such that*

$$\int \lambda(z)\pi(dz) = \frac{\alpha}{1-\beta\mathbb{E}^q[H(a)]}. \tag{3.4}$$

Proof. The ergodicity is a well known result for linear marked Hawkes process. Let π be the invariant probability measure for Z_t, then

$$\int \lambda(z)\pi(dz) = \alpha + \beta \int_{\mathbb{X}} \int_0^\infty h(t,a)dtq(da) \int \lambda(z)\pi(dz). \tag{3.5}$$

$\square$

Lemma 6 (Lower Bound)**.**

$$\liminf_{t\to\infty} \frac{1}{t}\log\mathbb{E}[e^{\theta N_t}] \geq \begin{cases} \nu(f(\theta)-1) & \text{if } \theta \in (-\infty,\theta_c] \\ +\infty & \text{otherwise} \end{cases}, \tag{3.6}$$

where $f(\theta)$ *is the minimal solution to* $x = \int e^{\theta + H(a)(x-1)}q(da)$ *and* θ_c *is defined in* (3.2).

Proof. For linear marked Hawkes process, the intensity at time t is $\lambda_t := \lambda(Z_t)$ where $\lambda(z) = \nu + z$ and $Z_t = \sum_{\tau_i<t} h(t-\tau_i, a_i)$. We tilt λ to $\hat{\lambda}$ and q to $\hat{q}$ such that by Girsanov formula the tilted probability measure $\hat{\mathbb{P}}$ is given by

$$\frac{d\hat{\mathbb{P}}}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = \exp\left\{\int_0^t (\lambda(Z_s)-\hat{\lambda}(Z_s))ds + \int_0^t \log\left(\frac{\hat{\lambda}(Z_s)}{\lambda(Z_s)}\right) + \log\left(\frac{d\hat{q}}{dq}\right) dN_s\right\}. \tag{3.7}$$

Let $\mathcal{Q}_e$ be the set of $(\hat{\lambda}, \hat{q}, \hat{\pi})$ such that the marked Hawkes process with intensity $\hat{\lambda}(Z_t)$ and random marks distributed as $\hat{q}$ is ergodic with $\hat{\pi}$ as the invariant measure of Z_t.

By ergodic theorem and Jensen's inequality, for any $(\hat{\lambda}, \hat{q}, \hat{\pi}) \in \mathcal{Q}_e$, we have,

$$\begin{aligned} &\liminf_{t\to\infty} \frac{1}{t}\log\mathbb{E}[e^{\theta N_t}] \qquad (3.8)\\ &\geq \liminf_{t\to\infty} \hat{\mathbb{E}}\left[\frac{1}{t}\theta N_t - \frac{1}{t}\int_0^t (\lambda-\hat{\lambda})ds - \frac{1}{t}\int_0^t \left[\log(\hat{\lambda}/\lambda) + \log(d\hat{q}/dq)\right]\hat{\lambda}ds\right] \\ &= \int \theta\hat{\lambda}\hat{\pi}(dz) + \int (\hat{\lambda}-\lambda)\hat{\pi}(dz) - \iint \left[\log(\hat{\lambda}/\lambda) + \log(d\hat{q}/dq)\right]\hat{\lambda}\hat{q}\hat{\pi}(dz). \end{aligned}$$

Hence, we have

$$\begin{aligned}
&\liminf_{t\to\infty}\frac{1}{t}\log\mathbb{E}[e^{\theta N_t}] \qquad (3.9)\\
&\geq \sup_{(\hat{\lambda},\hat{q},\hat{\pi})\in\mathcal{Q}_e}\left\{\int\theta\hat{\lambda}\hat{\pi}+\int(\hat{\lambda}-\lambda)\hat{\pi}-\iint\left[\log(\hat{\lambda}/\lambda)+\log(d\hat{q}/dq)\right]\hat{\lambda}\hat{q}\hat{\pi}\right\}.\\
&\geq \sup_{(K\lambda,\hat{q},\hat{\pi})\in\mathcal{Q}_e}\int\left[\left(\theta-\mathbb{E}^{\hat{q}}[\log(d\hat{q}/dq)]\right)\hat{\lambda}+\hat{\lambda}-\lambda-\hat{\lambda}\log\left(\hat{\lambda}/\lambda\right)\right]\hat{\pi}\\
&\geq \sup_{0<K<\mathbb{E}^{\hat{q}}[H(a)]^{-1},(K\lambda,\hat{q},\hat{\pi})\in\mathcal{Q}_e}\int\left[\left(\theta-\mathbb{E}^{\hat{q}}[\log(d\hat{q}/dq)]\right)+1-\frac{1}{K}-\log K\right]\hat{\lambda}\hat{\pi}\\
&=\sup_{\hat{q}}\sup_{0<K<\mathbb{E}^{\hat{q}}[H(a)]^{-1}}\left[\left(\theta-\mathbb{E}^{\hat{q}}[\log(d\hat{q}/dq)]\right)+1-\frac{1}{K}-\log K\right]\cdot\frac{K\nu}{1-K\mathbb{E}^{\hat{q}}[H(a)]},
\end{aligned}$$

where the last equality is obtained by applying Lemma 5. The supremum of $\hat{q}$ is taken over $\mathcal{M}(\mathbb{X})$ such that $\hat{q}$ is absolutely continuous w.r.t. q. Optimizing over $K>0$, we get

$$\begin{aligned}
&\liminf_{t\to\infty}\frac{1}{t}\log\mathbb{E}[e^{\theta N_t}] \qquad (3.10)\\
&\geq\begin{cases}\sup_{\hat{q}}\nu(\hat{f}(\theta)-1) & \text{if } \theta\in\left(-\infty,\mathbb{E}^{\hat{q}}\left[\log\frac{d\hat{q}}{dq}\right]+\mathbb{E}^{\hat{q}}[H(a)]-1-\log\mathbb{E}^{\hat{q}}[H(a)]\right]\\ +\infty & \text{otherwise}\end{cases},
\end{aligned}$$

where $\hat{f}(\theta)$ is the minimal solution to the equation

$$\begin{aligned}
x&=e^{\theta+\mathbb{E}^{\hat{q}}[\log(dq/d\hat{q})]+\mathbb{E}^{\hat{q}}[H(a)](x-1)} \qquad (3.11)\\
&\leq\mathbb{E}^{\hat{q}}\left[e^{\theta+H(a)(x-1)}\frac{dq}{d\hat{q}}\right]=\int e^{\theta+H(a)(x-1)}q(da).
\end{aligned}$$

The last inequality holds by applying Jensen's inequality and the equality holds if and only if

$$\frac{d\hat{q}}{dq}=\frac{e^{H(a)(x-1)}}{\mathbb{E}^{q}[e^{H(a)(x-1)}]}. \qquad (3.12)$$

Optimizing over $\hat{q}$, we get

$$\liminf_{t\to\infty}\frac{1}{t}\log\mathbb{E}[e^{\theta N_t}]\geq\begin{cases}\nu(f(\theta)-1) & \text{if } \theta\in(-\infty,\theta_c]\\ +\infty & \text{otherwise,}\end{cases} \qquad (3.13)$$

where θ_c is some critical value to be determined. Let

$$G(x)=e^{\theta}\int e^{H(a)(x-1)}q(da)-x. \qquad (3.14)$$

If $\theta=0$, then $G(x)=\int e^{H(a)(x-1)}q(da)-x$ satisfies $G(1)=0$, $G(\infty)=\infty$ (by (1.10)) and $G'(1)=\mathbb{E}^{q}[H(a)]-1<0$ which implies that $\min_{x>1}G(x)<0$. Hence, there exists some critical $\theta_c>0$ such that $\min_{x>1}G(x)=0$. The critical values x_c and θ_c satisfy $G(x_c)=G'(x_c)=0$, which imlies that

$$\theta_c=-\log\int H(a)e^{H(a)(x_c-1)}q(da), \qquad (3.15)$$

where $x_c>1$ satisfies the equation $x\int H(a)e^{H(a)(x-1)}q(da)=\int e^{H(a)(x-1)}q(da)$.

It is easy to check that indeed for $dq_* = \frac{e^{H(a)(x_*-1)}}{\mathbb{E}^q[e^{H(a)(x_*-1)}]}dq$,

$$\mathbb{E}^{q_*}\left[\log\frac{dq_*}{dq}\right] + \mathbb{E}^{q_*}[H(a)] - 1 - \log\mathbb{E}^{q_*}[H(a)]) = -\log\int H(a)e^{H(a)(x_*-1)}q(da). \tag{3.16}$$

□

Lemma 7 (Upper Bound)**.**

$$\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}[e^{\theta N_t}] \le \begin{cases}\nu(f(\theta)-1) & \text{if } \theta\in(-\infty,\theta_c] \\ +\infty & \text{otherwise}\end{cases}, \tag{3.17}$$

where $f(\theta)$ is the minimal solution to $x = \int e^{\theta+H(a)(x-1)}q(da)$ and θ_c is defined in (3.2)*.*

Proof. It is well known that linear Hawkes process has a immigration-birth representation. The immigrants (roots) arrive with usual Poisson process with constant intensity $\nu > 0$. Each immigrants then generate children according to the Galton-Watson structure. (See for example Karabash [8]) Consider a random rooted tree (with root, i.e. immigrant, at time 0) associated to Hawkes process via Galton-Watson interpretation. Note here root is unmarked at the start of the process so marking goes into expectation calculation later. Let K be the number of children of root node and $S_t^{(1)}, S_t^{(2)}, \dots, S_t^{(K)}$ be the number of descendants of root's k-th child that were born before time t (including k-th child if an only if it was born before time t). Let S_t be the total number of children in tree before time t including root node. Then

$$\begin{aligned}
F_S(t) &:= \mathbb{E}[\exp(\theta S_t)] && (3.18)\\
&= \sum_{k=0}^{\infty}\mathbb{E}[\exp(\theta S_t)|K=k]\mathbb{P}(K=k) && (3.19)\\
&= \exp(\theta)\sum_{k=0}^{\infty}\mathbb{P}(K=k)\prod_{i=1}^{k}\mathbb{E}\left[\exp\left(\theta S_t^{(i)}\right)\right] && (3.20)\\
&= \exp(\theta)\sum_{k=0}^{\infty}\mathbb{E}\left[\exp\left(\theta S_t^{(1)}\right)\right]^k\mathbb{P}(K=k) && (3.21)\\
&= \exp(\theta)\sum_{k=0}^{\infty}\int_{\mathbb{X}}\left[\left(\int_0^t\frac{h(s,a)}{H(a)}F_S(t-s)ds\right)^k e^{-H(a)}\frac{H(a)^k}{k!}\right]q(da) && (3.22)\\
&= \int_{\mathbb{X}}\exp\left(\theta+\int_0^t h(s,a)(F_S(t-s)-1)ds\right)q(da). && (3.23)
\end{aligned}$$

Now observe that $F_S(t)$ is strictly increasing and hence must approach to smallest solution x_* of the following equation

$$x = \int_{\mathbb{X}}\exp\left[\theta + H(a)(x-1)\right]q(da). \tag{3.24}$$

Finally, since random roots arrive according to a Poisson process with constant intensity $\nu > 0$, we have

$$F_N(t) := \mathbb{E}[\exp(\theta N_t)] = \exp\left[\nu\int_0^t(F_S(t-s)-1)ds\right]. \tag{3.25}$$

But since $F_S(s) \uparrow x_*$ as $s \to \infty$ we obtain the main result

$$\frac{1}{t}\log F_N(t) = \nu \frac{1}{t}\left[\int_0^t (F_S(s)-1)\,ds\right] \underset{t\to\infty}{\longrightarrow} \nu(x_*-1), \tag{3.26}$$

which proves the desired formula; note that $x_* = \infty$ when there is no solution to (3.24) and that completes the proof. □

3.2. **Large Deviation Principle.** In this section, we prove the main result, i.e. Theorem 1 by using Gärtner-Ellis theorem for the upper bound and tilting method for the lower bound.

Proof of Theorem 1. For the upper bound, since we have Theorem 4, we can simply apply Gärtner-Ellis theorem. To prove the lower bound, it suffices to show that for any $x > 0$, $\epsilon > 0$, we have

$$\liminf_{t\to\infty} \frac{1}{t}\log \mathbb{P}\left(\frac{N_t}{t} \in B_\epsilon(x)\right) \geq -\sup_{\theta\in\mathbb{R}}\{\theta x - \Gamma(\theta)\}, \tag{3.27}$$

where $B_\epsilon(x)$ denotes the open ball centered at x with radius ϵ. Let $\hat{\mathbb{P}}$ denote the tilted probability measure with rate $\hat{\lambda}$ and marks distributed as $\hat{q}(da)$ as defined in Theorem 6. By Jensen's inequality, we have

$$\begin{aligned}
&\frac{1}{t}\log \mathbb{P}\left(\frac{N_t}{t} \in B_\epsilon(x)\right) \qquad (3.28)\\
&\geq \frac{1}{t}\log \int_{\frac{N_t}{t}\in B_\epsilon(x)} \frac{d\mathbb{P}}{d\hat{\mathbb{P}}}d\hat{\mathbb{P}}\\
&= \frac{1}{t}\log \hat{\mathbb{P}}\left(\frac{N_t}{t} \in B_\epsilon(x)\right) - \frac{1}{t}\log\left[\frac{1}{\hat{\mathbb{P}}\left(\frac{N_t}{t}\in B_\epsilon(x)\right)}\int_{\frac{N_t}{t}\in B_\epsilon(x)} \frac{d\hat{\mathbb{P}}}{d\mathbb{P}}d\hat{\mathbb{P}}\right]\\
&\geq \frac{1}{t}\log \hat{\mathbb{P}}\left(\frac{N_t}{t} \in B_\epsilon(x)\right) - \frac{1}{\hat{\mathbb{P}}\left(\frac{N_t}{t}\in B_\epsilon(x)\right)}\cdot\frac{1}{t}\cdot\hat{\mathbb{E}}\left[1_{\frac{N_t}{t}\in B_\epsilon(x)}\log\frac{d\hat{\mathbb{P}}}{d\mathbb{P}}\right].
\end{aligned}$$

By ergodic theorem, we get

$$\liminf_{t\to\infty} \frac{1}{t}\log \mathbb{P}\left(\frac{N_t}{t} \in B_\epsilon(x)\right) \geq -\inf_{\substack{0<K<\mathbb{E}^{\hat{q}}[H(a)]^{-1}\\ (K\lambda,\hat{q},\hat{\pi})\in\mathcal{Q}_e^x}} \mathcal{H}(\hat{\lambda},\hat{q},\hat{\pi}). \tag{3.29}$$

where $\mathcal{Q}_e^x$ is defined as

$$\mathcal{Q}_e^x = \left\{(\hat{\lambda},\hat{q},\hat{\pi}) \in \mathcal{Q}_e : \int \hat{\lambda}(z)\hat{\pi}(dz) = x\right\}. \tag{3.30}$$

and the relative entropy $\mathcal{H}$ is

$$\mathcal{H}(\hat{\lambda},\hat{q},\hat{\pi}) = \int(\lambda-\hat{\lambda})\hat{\pi} + \int \log(\hat{\lambda}/\lambda)\hat{\lambda}\hat{\pi} + \iint \log(d\hat{q}/dq)\hat{q}\hat{\lambda}\hat{\pi}. \tag{3.31}$$

By Lemma 5, we have

$$\inf_{0<K<\mathbb{E}^{\hat{q}}[H(a)]^{-1},x=\frac{\nu K}{1-K\mathbb{E}^{\hat{q}}[H(a)]},(K\lambda,\hat{q},\hat{\pi})\in\mathcal{Q}_e}\mathcal{H}(\hat{\lambda},\hat{q},\hat{\pi}) \tag{3.32}$$
$$=\inf_{K=\frac{x}{x\mathbb{E}^{\hat{q}}[H(a)]+\nu},(K\lambda,\hat{q},\hat{\pi})\in\mathcal{Q}_e}\left\{\frac{1}{K}-1+\log K+\mathbb{E}^{\hat{q}}\left[\log\frac{d\hat{q}}{dq}\right]\right\}\int\hat{\lambda}\hat{\pi}$$
$$=\inf_{\hat{q}}\left\{\mathbb{E}^{\hat{q}}[H(a)]+\frac{\nu}{x}-1+\log\left(\frac{x}{x\mathbb{E}^{\hat{q}}[H(a)]+\nu}\right)+\mathbb{E}^{\hat{q}}\left[\log\frac{d\hat{q}}{dq}\right]\right\}x$$
$$=\inf_{\hat{q}}\left\{x\mathbb{E}^{\hat{q}}[H(a)]+\nu-x+x\log\left(\frac{x}{x\mathbb{E}^{\hat{q}}[H(a)]+\nu}\right)+x\mathbb{E}^{\hat{q}}\left[\log\frac{d\hat{q}}{dq}\right]\right\}.$$

Next, let us find a more explict form for the Legendre-Fenchel transform of $\Gamma(\theta)$.

$$\sup_{\theta\in\mathbb{R}}\{\theta x-\Gamma(\theta)\}=\sup_{\theta\in\mathbb{R}}\{\theta x-\nu(f(\theta)-1)\}, \tag{3.33}$$

where $f(\theta)=\mathbb{E}^q[e^{\theta+(f(\theta)-1)H(a)}]$. Thus

$$f'(\theta)=\mathbb{E}^q\left[(1+f'(\theta)H(a))e^{\theta+(f(\theta)-1)H(a)}\right]. \tag{3.34}$$

The optimal θ_* for (3.33) would satisfy $f'(\theta_*)=\frac{x}{\nu}$ and θ_* and $x_*=f(\theta_*)$ satisfy the following equations

$$\begin{cases}x_*=\mathbb{E}^q\left[e^{\theta_*+(x_*-1)H(a)}\right]\\ \frac{x}{\nu}=x_*+\frac{x}{\nu}\mathbb{E}^q\left[H(a)e^{\theta_*+(x_*-1)H(a)}\right]\end{cases}, \tag{3.35}$$

and $\sup_{\theta\in\mathbb{R}}\{\theta x-\Gamma(\theta)\}=\theta_*x-\nu(x_*-1)$.

On the other hand, letting $dq_*=\frac{e^{(x_*-1)H(a)}}{\mathbb{E}^q[e^{(x_*-1)H(a)}]}dq$, we have

$$\mathbb{E}^{q_*}[H(a)]=\frac{\mathbb{E}^q\left[e^{\theta_*+(x_*-1)H(a)}\right]}{\mathbb{E}^q\left[e^{(x_*-1)H(a)}\right]}=\frac{1}{x_*}-\frac{\nu}{x}, \tag{3.36}$$

and $\mathbb{E}^{q_*}[\log\frac{dq_*}{dq}]=(x_*-1)\mathbb{E}^{q_*}[H(a)]-\log\mathbb{E}^q[e^{(x_*-1)H(a)}]$, which implies that

$$\liminf_{t\to\infty}\frac{1}{t}\log\mathbb{P}\left(\frac{N_t}{t}\in B_\epsilon(x)\right) \tag{3.37}$$
$$\geq-\inf_{\hat{q}}\left\{x\mathbb{E}^{\hat{q}}[H(a)]+\nu-x+x\log\left(\frac{x}{x\mathbb{E}^{\hat{q}}[H(a)]+\nu}\right)+x\mathbb{E}^{\hat{q}}\left[\log\frac{d\hat{q}}{dq}\right]\right\}$$
$$\geq-\left\{x\mathbb{E}^{q_*}[H(a)]+\nu-x+x\log\left(\frac{x}{x\mathbb{E}^{q_*}[H(a)]+\nu}\right)+x\mathbb{E}^{q_*}\left[\log\frac{dq_*}{dq}\right]\right\}$$
$$=\theta_*x-\nu(x_*-1)=\sup_{\theta\in\mathbb{R}}\{\theta x-\Gamma(\theta)\}.$$

□

4. Risk Model with Marked Hawkes Claims Arrivals

We consider the following risk model for the surplus process R_t of an insurance portfolio,

$$R_t=u+\rho t-\sum_{i=1}^{N_t}C_i, \tag{4.1}$$

where $u>0$ is the initial reserve, $\rho>0$ is the constant premium and C_i are i.i.d. positive random variables with the common distribution $\mu(dC)$. C_i represents the

claim size at the ith arrival time, which are independent of N_t, a marked Hawkes process in this paper.

For any $u > 0$, let

$$\tau_u = \inf\{t > 0 : R_t \leq 0\}, \tag{4.2}$$

and denote the infinite and finite horizon ruin probabilities by

$$\psi(u) = \mathbb{P}(\tau_u < \infty), \quad \psi(u, uz) = \mathbb{P}(\tau_u \leq uz), \quad u, z > 0. \tag{4.3}$$

By the law of large numbers,

$$\lim_{t\to\infty} \frac{1}{t} \sum_{i=1}^{N_t} C_i = \frac{\mathbb{E}^{\mu}[C]\nu}{1 - \mathbb{E}^{q}[H(a)]}. \tag{4.4}$$

Therefore, to exclude the trivial case, we need to assume that

$$\frac{\mathbb{E}^{\mu}[C]\nu}{1 - \mathbb{E}^{q}[H(a)]} < \rho < \frac{\nu(x_c) - 1}{\theta_c}, \tag{4.5}$$

where the critical values θ_c and $x_c = f(\theta_c)$ satisfy

$$\begin{cases} x_c = \int_{\mathbb{R}^+} \int_{\mathbb{X}} e^{\theta_c C + H(a)(x_c - 1)} q(da)\mu(dC) \\ 1 = \int_{\mathbb{R}^+} \int_{\mathbb{X}} H(a) e^{H(a)(x_c-1)+\theta_c C} q(da)\mu(dC) \end{cases}. \tag{4.6}$$

We have (following the proofs of large deviation results in Section 3)

$$\Gamma_C(\theta) := \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}\left[e^{\theta \sum_{i=1}^{N_t} C_i}\right] = \begin{cases} \nu(x-1) & \text{if } \theta \in (-\infty, \theta_c] \\ +\infty & \text{otherwise} \end{cases}, \tag{4.7}$$

where x is the minimal solution to the equation

$$x = \int_{\mathbb{R}^+} \int_{\mathbb{X}} e^{\theta C + (x-1)H(a)} q(da)\mu(dC). \tag{4.8}$$

Before we proceed, let us quote a result from Glynn and Whitt [6], which will be used in our proof Theorem 10.

Theorem 8 (Glynn and Whitt [6])**.** *Let S_n be random variables. $\tau_u = \inf\{n : S_n > u\}$ and $\psi(u) = \mathbb{P}(\tau_u < \infty)$. Assume that there exist $\gamma, \epsilon > 0$ such that*
(i) $\kappa_n(\theta) = \log \mathbb{E}[e^{\theta S_n}]$ is well defined and finite for $\gamma - \epsilon < \theta < \gamma + \epsilon$.
(ii) $\limsup_{n\to\infty} \mathbb{E}[e^{\theta(S_n - S_{n-1})}] < \infty$ for $-\epsilon < \theta < \epsilon$.
(iii) $\kappa(\theta) = \lim_{n\to\infty} \frac{1}{n}\kappa_n(\theta)$ exists and is finite for $\gamma - \epsilon < \theta < \gamma + \epsilon$.
(iv) $\kappa(\gamma) = 0$ and κ is differentiable at γ with $0 < \kappa'(\gamma) < \infty$.
Then, $\lim_{u\to\infty} \frac{1}{u} \log \psi(u) = -\gamma$.

Remark 9. *We claim that $\Gamma_C(\theta) = \rho\theta$ has a unique positive solution $\theta^\dagger < \theta_c$. Let $G(\theta) = \Gamma_C(\theta) - \rho\theta$. Notice that $G(0) = 0$, $G(\infty) = \infty$ and G is convex. We also have $G'(0) = \frac{\mathbb{E}^{\mu}[C]\nu}{1-\mathbb{E}^{q}[H(a)]} - \rho < 0$ and $\Gamma_C(\theta_c) - \rho\theta_c > 0$ since we assume that $\rho < \frac{\nu(f(\theta_c)-1)}{\theta_c}$. Therefore, there exists unique $\theta^\dagger \in (0, \theta_c)$ such that $\Gamma_C(\theta^\dagger) = \rho\theta^\dagger$.*

Theorem 10 (Infinite Horizon)**.** *Assume all the assumptions in Theorem 1 and in addition (4.5), we have $\lim_{u\to\infty} \frac{1}{u} \log \psi(u) = -\theta^\dagger$, where $\theta^\dagger \in (0, \theta_c)$ is the unique positive solution of $\Gamma_C(\theta) = \rho\theta$.*

Proof. Take $S_t = \sum_{i=1}^{N_t} C_i - \rho t$ and $\kappa_t(\theta) = \log \mathbb{E}[e^{\theta S_t}]$. Then $\lim_{t\to\infty} \frac{1}{t}\kappa_t(\theta) = \Gamma_C(\theta) - \rho\theta$. Consider $\{S_{nh}\}_{n\in\mathbb{N}}$, we have $\lim_{n\to\infty} \frac{1}{n}\kappa_{nh}(\theta) = h\Gamma_C(\theta) - h\rho\theta$. Checking the conditions in Theorem 8 and applying it, we get

$$\lim_{u\to\infty} \frac{1}{u} \log \mathbb{P}\left(\sup_{n\in\mathbb{N}} S_{nh} > u\right) = -\theta^\dagger. \tag{4.9}$$

Finally, notice that

$$\sup_{t\in\mathbb{R}^+} S_t \geq \sup_{n\in\mathbb{N}} S_{nh} \geq \sup_{t\in\mathbb{R}^+} S_t - \rho h. \tag{4.10}$$

Hence, $\lim_{u\to\infty} \frac{1}{u} \log \psi(u) = -\theta^\dagger$. □

Theorem 11 (Finite Horizon)**.** *Under the same assumptions as in Theorem 10, we have*

$$\lim_{u\to\infty} \frac{1}{u} \log \psi(u, uz) = -w(z), \quad \textit{for any } z > 0, \tag{4.11}$$

where

$$w(z) = \begin{cases} z\Lambda_C\left(\frac{1}{z} + \rho\right) & \textit{if } 0 < z < \frac{1}{\Gamma'(\theta^\dagger) - \rho} \\ \theta^\dagger & \textit{if } z \geq \frac{1}{\Gamma'(\theta^\dagger)-\rho} \end{cases}, \tag{4.12}$$

where $\Lambda_C(x) = \sup_{\theta\in\mathbb{R}}\{\theta x - \Gamma_C(\theta)\}$ *and* $\theta^\dagger \in (0, \theta_c)$ *is the unique positive solution of* $\Gamma_C(\theta) = \rho\theta$*.*

Proof. The proof is similar as in Stabile and Torrisi [9] and we omit it here. □

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